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NORM ATTAINING OPERATORS FROM L_1 INTO L_∞

BY

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ABSTRACT

We show that the set of norm attaining operators is dense in the space of all bounded linear operators from L_1 into L_∞.

A bounded linear operator T between Banach spaces X and Y attains its norm if there is $x_0 \in B_X$ (the closed unit ball of X) such that

$$\|Tx_0\| = \|T\| := \sup\{\|Tx\|: x \in B_X\}.$$

Starting from the Bishop–Phelps Theorem [3], that the set of norm attaining linear functionals on a Banach space X is dense in the dual space X^*, a lot of attention has been paid to the question if the set $NA(X,Y)$ of norm attaining operators is dense in the space $L(X,Y)$ of all bounded linear operators. We refer the reader to [10], [4], and [8] for details and background. In this note we settle the denseness problem for a concrete pair of classical Banach spaces by proving the following

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THEOREM: *For every σ-finite measure μ, the set of norm attaining operators from $L_1(\mu)$ into $L_\infty[0,1]$ is dense in the space $L(L_1(\mu), L_\infty[0,1])$ of all bounded linear operators.*

There is a relation to recent efforts to discuss the Bishop–Phelps Theorem in the bilinear context, which actually motivated our work. Recall that the space of continuous bilinear forms on a Banach space X is isometric to $L(X, X^*)$, the operator T which corresponds to a bilinear form φ being given by

$$\langle Tx, y\rangle = \varphi(x,y) \quad (x, y \in X).$$

If φ attains its norm, i.e. there are $x_0, y_0 \in B_X$ such that

$$|\varphi(x_0, y_0)| = \|\varphi\| := \sup\{|\varphi(x,y)| \colon x, y \in B_X\},$$

then it is clear that also the operator T attains its norm, but the converse is far from true. Y. S. Choi [5] has recently shown that the set of norm attaining bilinear forms on $L_1[0,1]$ is not dense. Thus, the above theorem shows that $L_1[0,1]$ is an example of a Banach space X such that $NA(X, X^*)$ is dense in $L(X, X^*)$, yet there are continuous bilinear forms on X that cannot be approximated by norm attaining forms. This answers a question posed in [2]. Further discussion of norm attaining multilinear mappings and polynomials can be found in [1], [6], and [9].

Our result may have the independent interest of giving a new example of a compact Hausdorff space K such that $NA(L_1[0,1], C(K))$ is dense in $L(L_1[0,1], C(K))$. The problem of characterizing compact spaces with this property was posed by W. Schachermayer [11] and remains open. It follows from results by J. Lindenstrauss [10] that K having a dense set of isolated points is a sufficient condition. Since $L_\infty[0,1]$ is isometric to $C(K)$ where K has no isolated points, that condition is far from necessary.

Proof of the Theorem: Let μ be a σ-finite measure on a set Ω and let m denote Lebesgue's measure on $I = [0,1]$. We start by recalling a well knownrepresentation of the space $L(L_1(\mu), L_\infty(m))$, which is nothing but $L_\infty(\mu \otimes m)$ where $\mu \otimes m$ denotes the product measure on $\Omega \times I$. More precisely, the operator $\hat{h}$ corresponding to an essentially bounded function h is given by

$$[\hat{h}(f)](t) = \int_\Omega h(\omega, t) f(\omega) \mathrm{d}\mu(\omega)$$

for almost every $t \in I$ and all $f \in L_1(\mu)$ (see [7, §3.2 and Ex. 3.27], for example).

Now let $h \in L_\infty(\mu \otimes m)$ be a simple function, i.e. a linear combination of characteristic functions of measurable subsets of $\Omega \times I$. The set of these simple functions is a dense subspace of $L_\infty(\mu \otimes m)$, so we are left with showing that the operator $\hat{h}$ corresponding to h attains its norm. By normalizing h and up to a rotation, we may assume without loss of generality that $\|h\|_\infty = 1$ and that h is identically 1 on a measurable set $S \subset \Omega \times I$ with positive measure.

For every $\omega \in \Omega$, the integral

$$F_\omega(x) = \int_0^x h(\omega, t)\mathrm{d}t \quad (0 \le x \le 1)$$

is differentiable, and its derivative is $h(\omega, x)$ a.e. in I. Since S has positive measure, there is a measurable set $A \subset \Omega$, with $0 < \mu(A) < \infty$, and $0 < y < 1$, so that $(\omega, y) \in S$ and so that $F'_\omega(y)$ exists, and is equal to $h(\omega, y) = 1$ for every $\omega \in A$. Taking $\delta_n \to 0$ and $B_n = [y - \delta_n, y + \delta_n]$, we obtain that $\|\mu(A)^{-1}\chi_A\|_1 = 1$ and that

$$\begin{aligned}\|\hat{h}(\mu(A)^{-1}\chi_A)\| &\ge \langle \hat{h}(\mu(A)^{-1}\chi_A), (1/2\delta_n)\chi_{B_n}\rangle \\ &= \frac{1}{\mu(A)}\int_A \frac{1}{2\delta_n}\int_{-\delta_n}^{\delta_n} h(\omega, y+t)\mathrm{d}t\mathrm{d}\mu(\omega) \longrightarrow 1\end{aligned}$$

as $n \to \infty$ by Lebesgue's dominated convergence theorem and by the choice of y. This shows that the operator $\hat{h}$ attains its norm at the function $\mu(A)^{-1}\chi_A$, as required. ∎

Let us point out a characterization of norm attaining operators which is suggested by the above proof. For a simple function h we got measurable sets $A \subset \Omega$, $0 < \mu(A) < \infty$, and $B_n \subset I$, $m(B_n) > 0$ for all n, so that

(1) $$\lim_{n\to\infty} \frac{1}{\mu(A)m(B_n)}\left|\int_{A\times B_n} h\mathrm{d}(\mu \otimes m)\right| = \|h\|_\infty.$$

Given an arbitrary function $h \in L_\infty(\mu \otimes m)$, condition (1) clearly implies that the operator $\hat{h}$ attains its norm. Let us see that (1) is also necessary in the real case. Assume that $\|h\|_\infty = 1$ and that $\hat{h}$ attains its norm. We first claim that there is a measurable set $A \subset \Omega$, $0 < \mu(A) < \infty$, so that $\hat{h}$ attains its norm at the function $\mu(A)^{-1}\chi_A$. Indeed, assume that $\|f\|_1 = 1 = \|\hat{h}(f)\|_\infty$, and use the Hahn–Banach Theorem to find $\Phi \in L_\infty[0,1]^*$ satisfying

$$\|\Phi\| = 1 = \langle \Phi, \hat{h}(f)\rangle = \langle \hat{h}^*(\Phi), f\rangle.$$

Then $\hat{h}^*(\Phi)$ is a norm attaining functional on $L_1(\mu)$, i.e. a function in $L_\infty(\mu)$ that attains its norm on a set of positive measure, and we simply take A to be a subset where the function is identically 1 or -1. Since $\|g\|_\infty = 1$ if and only if $\|g\|_\infty \leq 1$ and there are sets B_n so that $m(B_n)^{-1}|\int_{B_n} g| \to 1$, taking $g = \hat{h}(\mu(A)^{-1}\chi_A)$ and writing everything in terms of the kernel h gives (1).

In the complex case an obvious modification of the above argument only gives

$$\lim_{n\to\infty} \frac{1}{\mu(A)m(B_n)} \int_{A\times B_n} h(\omega,t)\theta(\omega)\mathrm{d}m(t)\mathrm{d}\mu(\omega) = \|h\|_\infty \tag{2}$$

for some measurable function θ whose modulus is identically 1. Nevertheless, (2) still implies that $\hat{h}$ attains its norm at the function $\mu(A)^{-1}\theta\chi_A$ and we have a characterization as well. To see that (2) may be strictly weaker than (1), it is enough to take $\mu = m$ and $h(\omega,t) = \exp(2\pi i\omega)$.

Finally, it is worth mentioning the striking contrast between $NA(L_1[0,1], L_\infty[0,1])$ and norm attaining bilinear forms on $L_1[0,1]$. Y. S. Choi [5] proved that there is a measurable subset S of the unit square such that the bilinear form corresponding to χ_S cannot be approximated by norm attaining bilinear forms. On the other hand, the operator corresponding to any simple function does attain its norm.

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References

[1] M. D. Acosta, F. J. Aguirre and R. Payá, *There is no bilinear Bishop–Phelps Theorem*, Israel Journal of Mathematics **93** (1996), 221–227.

[2] R. M. Aron, C. Finet and E. Werner, *Norm-attaining n-linear forms and the Radon–Nikodým property*, in *Proceedings of the 2^{nd} Conference on Function Spaces* (K. Jarosz, ed.), Lecture Notes in Pure and Applied Mathematics, Marcel Dekker, New York, 1995, pp. 19–28.

[3] E. Bishop and R. R. Phelps, *A proof that every Banach space is subreflexive*, Bulletin of the American Mathematical Society **67** (1961), 97–98.

[4] J. Bourgain, *On dentability and the Bishop–Phelps property*, Israel Journal of Mathematics **28** (1977), 265–271.

[5] Y. S. Choi, *Norm attaining bilinear forms in* $L^1[0,1]$, Journal of Mathematical Analysis and Applications **211** (1997), 295–300.

[6] Y. S. Choi and S. G. Kim, *Norm or numerical radius attaining multilinear mappings and polynomials*, Journal of the London Mathematical Society **54** (1996), 135–147.

[7] A. Defant and K. Floret, *Tensor Norms and Operator Ideals*, North-Holland Mathematics Studies **176**, Elsevier, Amsterdam, 1993.

[8] W. T. Gowers, *Symmetric block bases of sequences with large average growth*, Israel Journal of Mathematics **69** (1990), 129–151.

[9] M. Jiménez-Sevilla and R. Payá, *Norm attaining multilinear forms and polynomials on preduals of Lorentz sequence spaces*, Studia Mathematica **127** (1998), 99–112.

[10] J. Lindenstrauss, *On operators which attain their norm*, Israel Journal of Mathematics **1** (1963), 139–148.

[11] W. Schachermayer, *Norm attaining operators on some classical Banach spaces*, Pacific Journal of Mathematics **105** (1983), 427–438.